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Double-shearing theory applied to instability and strain localization in granular materials

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Abstract. An account is given of the non-dilatant double-shearing theory of plane flow of granular materials, and it is shown that the theory may be formulated as a special form of hypoplasticity theory. It is shown that according to this theory, simple shearing flows may be supported by a time-independent stress field, but that this solution is unstable. An alternative solution in which the stress in time-dependent is also derived, and shear flow takes place under decreasing shear stress. The strain localization theory of Rudnicki and Rice is applied in conjunction with the double-shearing theory, and it is shown that the theory admits bifurcations in which shear bands form on planes that coincide with the shear plane. Similarly, in pure shear, there exists an unstable solution with time-independent stress, and a solution with time-dependent stress in which the compressive load falls as the deformation increases, and shear bands may form at surfaces on which, according to the Coulomb criterion, the critical shear stress is mobilized. The double-shearing theory for axially symmetric flow is summarized, and applied to compression of a circular cylinder. Again there is an unstable constant stress solution, a time-dependent stress solution in which the axial pressure decreases as the compression of the cylinder increases, and conical shear bands may form on conical surfaces on which the critical shear stress is mobilized.

Key words: granular flow, instability, pure shear, shear bands, simple shear, triaxial test

1. Introduction

In this paper we explore some consequences of the non-dilalant double-shearing theory of the mechanical behaviour of granular materials. This theory was formulated by Spencer [1, 2], and has had some success in describing problems of flow of granular materials as, for example in [3, 4, 5]. The formulation of constitutive equations for the mechanics of granular materials is still a matter of debate. Many different models have been proposed, but none has found general acceptance The literature is too extensive to be summarized here, but a brief review up to 1982 was given in [1], and some discussions of recent relevant work have been given by, among others, Collins [6], Harris [7] and Nemat-Nasser [8] There are many other reviews which approach the subject from various viewpoints. One purpose of the present study is to obtain results that allow comparison with those of other theories and with experiment.

The double-shearing theory is based on the Coulomb failure criterion, supplemented by a kinematic constitutive assumption that the deformation mechanism is by simultaneous shearing on the two families of surfaces on which the critical shear stress is mobilized. A description of the plane strain theory follows in Section 2. The theory can be extended in various ways. In particular, Mehrabadi and Cowin [9, 10, 11] have proposed a 'dilatant double-shearing theory' in which shearing is accompanied by an expansion in the direction normal to the shear plane. However, in this paper we restrict attention to the simpler non-dilatant theory, which we believe is sufficient to capture many, though not all, of the features of flows of granular materials.

It is well known that uniform flows of many real granular materials are difficult to realize in practice, and that they tend to become unstable. These instabilities are often manifested by the formation of shear bands, which are narrow zones of intense shearing deformation. It has also been pointed out by several authors (for example, Harris [12]) that the double-shearing equations (and several other models of granular material mechanics) are linearly ill-posed in the sense that small perturbations of solutions of the equations may grow exponentially. Schaeffer [13] suggested that this property of the equations may be associated with shear-band formation.

In this paper we apply the strain localization analysis formulated by Rudnicki and Rice [14] and Rice [15] to shear-band formation in granular material using the double-shearing theory. This analysis has been applied using various constitutive equations but not, as far as we are aware, in conjunction with the double-shearing theory. We consider the stability of simple shearing, pure shearing, and triaxial compression flows. It is shown that the theory admits shear-band instabilities that are broadly in agreement with observations of these flows.

Because accounts of the formulation of the double-shearing theory are not easily accessible, we describe the plane strain theory in some detail in Section 2. The description given in Section 2 differs a little from the formulations given in [1, 2]. The earlier formulation is also extended by expressing the double-shearing equations in a new form that shows them to be a special form of hypoplastic constitutive equation.

Simple shearing flow is discussed in Section 3. It is shown that there exists a time-independent uniform stress that is compatible with a simple shearing flow, but that this solution is linearly unstable. It is also shown that simple shearing flow can accompany a time-dependent stress field in which the principal axes of stress rotate *away* from their directions in the steady stress solution, and that shear flow the occurs under a *decreasing* shear stress, which also indicates unstable behaviour. The strain localization analysis of Rudnicki and Rice [14] and Rice [15] is applied to the plane double-shearing theory, and it is shown that shear bands may form in surfaces on which the critical shear stress is mobilized. In the case of simple shear, one such family of surfaces coincides with the shear planes, and so shear bands may form on the shear planes. This is broadly in line with observation.

Pure shear is treated in a similar way in Section 4. In this case also there is a time-independent stress solution which is shown to be linearly unstable, and a time-dependent stress solution in which the principal stress axes rotate away from their directions in the steady-stress solution, and compression takes place under a decreasing load. In this case the shear band analysis predicts formation of shear bands at angles $\pm(\frac{1}{4}\pi + \frac{1}{2}\phi)$ to the horizontal (where ϕ is the angle of internal friction), which is also in broad agreement with observation.

In Section 5 we outline the double-shearing theory for axially symmetric deformations, and in Section 6 apply the theory to the problem of compression of a circular cylinder (the triaxial test of soil mechanics). Results are very similar to those of the pure shear problem. There is a linearly unstable time-independent stress solution, and a time-dependent stress solution in which flow occurs under decreasing axial compressive stress. The shear-band analysis admits the formation of conical shear bands in conical surfaces whose walls have slope $\pm(\frac{1}{4}\pi + \frac{1}{2}\phi)$.

2. General theory-plane strain

Initially all quantities are referred to a fixed system of rectangular Cartesian coordinates *Oxyz*. The components of the stress tensor σ are denoted as

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}, \tag{2.1}$$

and the components of the velocity **v** by (u, v, w). In the first instance we consider plane strain in the (x, z) planes, so that $v = 0$, u and w are functions of x and z, and the relevant stress components are σ_{xx}, σ_{xz}, and σ_{zz}, all of which depend only on x and z. We write

$$p = -\frac{1}{2}(\sigma_{xx} + \sigma_{zz}), \quad q = \left\{\frac{1}{4}(\sigma_{xx} - \sigma_{zz})^2 + \sigma_{xz}^2\right\}^{\frac{1}{2}}, \quad q \geq 0, \tag{2.2}$$

so that p and q are stress invariants that represent the mean in-plane hydrostatic pressure and the maximum shear stress, respectively. The stress angle ψ is defined by

$$\tan 2\psi = \frac{2\sigma_{xz}}{\sigma_{xx} - \sigma_{zz}}, \tag{2.3}$$

and is the angle that the principal stress axis associated with the algebraically greater principal stress makes with the x-axis (tensile stress is taken to be positive). Then the relevant stress components can be expressed as

$$\sigma_{xx} = -p + q\cos 2\psi, \quad \sigma_{zz} = -p - q\cos 2\psi, \quad \sigma_{xz} = q\sin 2\psi. \tag{2.4}$$

In soil mechanics terminology, the case $\cos 2\psi > 0$ corresponds to passive lateral pressure and $\cos 2\psi < 0$ corresponds to active lateral pressure.

The material is assumed to conform to the Coulomb-Mohr yield condition

$$q \leq p\sin\phi + c\cos\phi \tag{2.5}$$

where ϕ is the angle of internal friction and c is the cohesion, both of which are assumed to be constant, and (2.5) holds as an equality whenever the material is undergoing deformation. In physical terms, (2.5) states that flow can only take place when the maximum shear stress q reaches the critical value $p\sin\phi + c\cos\phi$. The argument leading to (2.5) (which essentially was stated by Coulomb [16]) is as follows. Consider an arbitrary curve in the (x, z) plane with slope $\tan\gamma$ and therefore normal unit vector $(-\sin\gamma, \cos\gamma)$. Then for a given stress, the normal compressive component σ and the tangential component τ of the traction on this surface are

$$\begin{aligned} \sigma &= \sigma_{xx}\sin^2\gamma - 2\sigma_{xz}\sin\gamma\cos\gamma + \sigma_{zz}\cos^2\gamma, \\ \tau &= -(\sigma_{xx} - \sigma_{zz})\sin\gamma\cos\gamma + \sigma_{xz}(\cos^2\gamma - \sin^2\gamma), \end{aligned} \tag{2.6}$$

which, using (2.4), can be expressed as

$$\sigma = -p + q\cos 2(\gamma - \psi), \quad \tau = q\sin 2(\gamma - \psi). \tag{2.7}$$

The Coulomb assumption is that $|\tau| \leq \sigma\tan\phi + c$ on every surface, and that flow can only occur when $|\tau| = \sigma\tan\phi + c$ on some surface. From (2.7)

$$|\tau| - \sigma \tan\phi = -p\tan\phi + q\sec\phi\sin\{2\,|\gamma - \psi| - \phi\} \tag{2.8}$$

and for a given stress this takes its maximum value when the surface is such that $\gamma = \psi \pm (\frac{1}{2}\phi + \frac{1}{4}\pi)$. It follows that the shear traction is critical when $q = p\sin\phi + c\cos\phi$, and that this critical traction is mobilized on the surfaces

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \tan\left(\psi \pm (\frac{1}{4}\pi + \frac{1}{2}\phi)\right). \tag{2.9}$$

This argument uses only standard properties of the stress tensor, and is quite independent of any considerations of equilibrium or momentum.

The equations of motion are, in plane strain and neglecting body forces

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xz}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right), \frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{zz}}{\partial z} = \rho\left(+\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right). \tag{2.10}$$

In quasi-static flows, when inertia terms are neglected, the Coulomb-Mohr condition (2.5) (as an equality) and the equilibrium equations (*i.e.* (2.10) with the right-hand sides set to zero) can be expressed as a pair of first-order partial differential equations for q and ψ; these equations are hyperbolic with (2.9) as their characteristics. However, this property does not obtain in the dynamic case.

To complete the material description it is necessary to specify a 'flow rule' that relates the stress to the deformation. Whereas (2.5) is generally accepted as a reasonable constitutive assumption for the description of stress in dry granular materials, the formulation of an appropriate flow rule is still controversial. Many proposals have been made, some of which are compared in, for example, [2], [6] and [7]. In this paper we adopt the 'non-dilatant double-shearing' model [1, 2] which has a number of attractive features. Because the formulation of the double-shearing theory is not now easily accessible, and the basis of the theory has been misunderstood in the literature, we give an outline of the formulation that is slightly different from the formulations given in [1, 2].

The essential constitutive assumption is an extension of Coulomb's argument, leading to (2.5), that flow is possible when the critical shear stress is mobilized on the surfaces (2.9). This is extended by proposing that when deformation occurs it does so by simultaneous shears on the surfaces (2.9) in the directions tangential to these curves. By way of introduction and for motivation we first consider a body in a uniform state of stress that satisfies the Coulomb condition (2.5) as an equality, and suppose that the material undergoes a uniform single shear in the (x, z) plane on one of the families of surfaces defined by the curves (2.6); for definiteness say on the curves

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \tan(\psi - \frac{1}{4}\pi - \frac{1}{2}\phi). \tag{2.11}$$

Since the stress is a uniform stress, ψ is constant and the curves (2.11) are parallel straight lines. A uniform shear flow on these lines corresponds to the velocity field

$$\begin{aligned} u &= a[(x - x_0)\cos(2\psi - \phi) + (z - z_0)\{1 + \sin(2\psi - \phi)\}], \\ w &= a[-(x - x_0)\{1 - \sin(2\psi - \phi)\} - (z - z_0)\cos(2\psi - \phi)]. \end{aligned} \tag{2.12}$$

In this motion the direction of the velocity is parallel to the lines (2.11), and the magnitude of the velocity at (x, z) is proportional to the normal distance from (x, z) to the line that passes through (x_0, z_0). The constant $2a$ is the magnitude of the shear strain-rate. Similarly, for a single shear flow of magnitude $2b$ on the surfaces defined by the lines $\mathrm{d}z/\mathrm{d}x = \tan(\psi + \frac{1}{4}\pi + \frac{1}{2}\phi)$, the velocity field is

$$\begin{aligned} u &= b[-(x - x_0)\cos(2\psi + \phi) + (z - z_0)\{1 - \sin(2\psi + \phi)\}], \\ w &= b[-(x - x_0)\{1 + \sin(2\psi + \phi)\} + (z - z_0)\cos(2\psi + \phi)]. \end{aligned} \tag{2.13}$$

Unless boundary conditions dictate otherwise, there is no reason to give precedence to either family of shear surfaces, so the general homogeneous shearing flow is a superposition of the velocity fields (2.12) and (2.13).

If the stress is not uniform, so that ψ depends on x and z, then (2.12) and (2.13) are no longer valid, but we may still write, for shears on the critical lines (2.11) in the neighbourhood of (x_0, z_0)

$$\begin{aligned} \mathrm{d}u &= a[\mathrm{d}x\cos(2\psi - \phi) + \mathrm{d}z\{1 + \sin(2\psi - \phi)\}], \\ \mathrm{d}w &= a[-\mathrm{d}x\{1 - \sin(2\psi - \phi)\} - \mathrm{d}z\cos(2\psi - \phi)] \end{aligned} \tag{2.14}$$

and for shears on the lines $\mathrm{d}z/\mathrm{d}x = \tan(\psi + \frac{1}{4}\pi + \frac{1}{2}\phi)$

$$\begin{aligned} \mathrm{d}u &= b[-\mathrm{d}x\cos(2\psi + \phi) + \mathrm{d}z\{1 - \sin(2\psi + \phi)\}], \\ \mathrm{d}w &= b[-\mathrm{d}x\{1 + \sin(2\psi + \phi)\} + \mathrm{d}z\cos(2\psi + \phi)]. \end{aligned} \tag{2.15}$$

In the general case in which ψ depends on x and z, a single shearing deformation on one of the families (2.9) is not compatible with an isochoric deformation. The exceptions are when the critical curves are parallel straight lines or concentric circles. In all other cases, the velocity gradients implied by (2.14) or (2.15) do not satisfy the compatibility conditions

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right), \quad \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right). \tag{2.16}$$

However, it is possible for the double shearing deformations obtained by superposing (2.14) and (2.15) to be realized in an isochoric motion. For the superposed deformation, (2.14) and (2.15) give

$$\begin{aligned} \frac{\partial u}{\partial x} &= a\cos(2\psi - \phi) - b\cos(2\psi + \phi), \\ \frac{\partial u}{\partial z} &= a\{1 + \sin(2\psi - \phi)\} + b\{1 - \sin(2\psi + \phi)\}, \\ \frac{\partial w}{\partial x} &= -a\{1 - \sin(2\psi - \phi)\} - b\{1 + \sin(2\psi + \phi)\}, \\ \frac{\partial w}{\partial z} &= -a\cos(2\psi - \phi) + b\cos(2\psi + \phi). \end{aligned} \tag{2.17}$$

By eliminating a and b from (2.17) we can show that

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{2.18}$$

$$\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\cos 2\psi - \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right)\sin 2\psi + \sin\phi\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0. \tag{2.19}$$

Equation (2.18) is the condition that the flow is isochoric, and (2.19) expresses the condition that the flow consists of simultaneous shears on the critical surfaces. However, (2.19) is not complete, and is not acceptable as a constitutive equation because it does not satisfy the requirement of invariance under superposed rigid body rotations. The reason is that the directions of the critical curves are given by the angle ψ which describes the directions of the principal stress axes, and in general this angle varies in space and in time.. Hence the critical curves are not material curves, but are in motion relative to the material. Therefore, in the neighbourhood of a generic particle at (x_0, z_0) the deformation should be referred to a frame of reference in which the direction of the principal stress axes are fixed. Equivalently, we may refer the motion to the fixed (x, z) axes but replace the (clockwise) material spin $\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$ by $\dot{\psi} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$, where the superposed dot represents a material time derivative, so that $\dot{\psi}$ is the (anticlockwise) spin of the principal axes of stress through the particle at (x_0, z_0). Hence (2.19) is replaced by

$$\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\cos 2\psi - \left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right)\sin 2\psi + \sin\phi\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} + 2\Omega\right) = 0, \tag{2.20}$$

where

$$\Omega = \dot{\psi} = \frac{\partial \psi}{\partial t} + u\frac{\partial \psi}{\partial x} + w\frac{\partial \psi}{\partial z} \tag{2.21}$$

is the spin of the principal stress axes through a generic particle. With this addition (2.20) has the required invariance property and is a possible constitutive equation for an isotropic material. Because Ω appears in the equations, the formulation involves the stress-rate as well as the stress and the velocity gradients. Alternative derivations of (2.18) and (2.20) were given in [2]. If appropriate stress and velocity boundary conditions are established the above stress and velocity equations represent a complete set of equations for the description of plane granular flow of a granular material.

It is emphasized that the argument leading to (2.18) and (2.20) is purely a kinematic one, based on a single constitutive assumption, and is quite independent of eqiulibrium or momentum considerations. It can be shown if ψ is regarded as a known quantity, then (2.18) and (2.20) form a pair of hyperbolic equations for u and w whose characteristics coincide with (2.9), which are also the characteristics of the stress equations in the case of quasi-static deformations. This is a *consequence* of the constitutive assumption of the double-shearing deformation mechanism, and has not been introduced as a postulate. This coincidence of characteristics does not apply in the dynamic situation (in fact the dynamic equations are not hyperbolic) but this does not at all invalidate the purely kinematic constitutive assumption that is represented mathematically by (2.20).

For some of the analysis of this paper, it is useful to cast the governing equations in an alternative form. From (2.2) and (2.3) it follows that

$$\Omega = \dot{\psi} = \frac{(\sigma_{xx} - \sigma_{zz})\dot{\sigma}_{xz} - \sigma_{xz}(\dot{\sigma}_{xx} - \dot{\sigma}_{zz})}{4q^2}, \tag{2.22}$$

where superposed dots denote material time derivatives. Also, (2.20) may be written as

$$(\sigma_{xx} - \sigma_{zz})d_{xz} - \sigma_{xz}(d_{xx} - d_{zz}) + 2q(\omega_{xz} + \Omega)\sin\phi = 0 \tag{2.23}$$

where

$$d_{xx} = \frac{\partial u}{\partial x}, \quad d_{zz} = \frac{\partial w}{\partial z}, \quad d_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad \omega_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right). \tag{2.24}$$

Hence, by inserting (2.22) in (2.23)

$$(\sigma_{xx} - \sigma_{zz})(2qd_{xz} + \dot{\sigma}_{xz}\sin\phi) - \sigma_{xz}\{2q(d_{xx} - d_{zz}) + (\dot{\sigma}_{xx} - \dot{\sigma}_{zz})\sin\phi\} + 4q^2\omega_{xz}\sin\phi = 0 \tag{2.25}$$

or, using (2.2)

$$\begin{aligned} &(\sigma_{xx} - \sigma_{zz})[2qd_{xz} + \{\dot{\sigma}_{xz} + (\sigma_{xx} - \sigma_{zz})\omega_{xz}\}\sin\phi] \\ &-\sigma_{xz}[2q(d_{xx} - d_{zz}) + \{(\dot{\sigma}_{xx} - \dot{\sigma}_{zz}) - 4\sigma_{xz}\omega_{xz}\}\sin\phi] = 0, \end{aligned} \tag{2.26}$$

We may rearrange this equation in the form

$$\frac{2q(d_{xx} - d_{zz}) + \sin\phi\{(\dot{\sigma}_{xx} - \dot{\sigma}_{zz}) - 4\sigma_{xz}\omega_{xz}\}}{2q(\sigma_{xx} - \sigma_{zz})} = \frac{2qd_{xz} + \sin\phi\{\dot{\sigma}_{xz} + (\sigma_{xx} - \sigma_{zz})\omega_{xz}\}}{2q\sigma_{xz}}. \tag{2.27}$$

By setting each side of (2.27) equal to a parameter $\dot{\lambda}$, and using the isochoric condition in the form $d_{xx} + d_{zz} = 0$, it follows that the rate-of-deformation can be expressed as

$$\begin{aligned} d_{xx} &= -d_{zz} = \frac{1}{2}\dot{\lambda}(\sigma_{xx} - \sigma_{zz}) - \frac{1}{4q}\sin\phi\{(\dot{\sigma}_{xx} - \dot{\sigma}_{zz}) - 4\sigma_{xz}\omega_{xz}\}, \\ d_{xz} &= \dot{\lambda}\sigma_{xz} - \frac{1}{2q}\sin\phi\{\dot{\sigma}_{xz} + (\sigma_{xx} - \sigma_{zz})\omega_{xz}\}. \end{aligned} \tag{2.28}$$

The derivative

$$\overset{\triangledown}{\sigma} = \begin{bmatrix} \overset{\triangledown}{\sigma}_{xx} & \overset{\triangledown}{\sigma}_{xz} \\ \overset{\triangledown}{\sigma}_{xz} & \overset{\triangledown}{\sigma}_{zz} \end{bmatrix} = \begin{bmatrix} \dot{\sigma}_{xx} - 2\sigma_{xz}\omega_{xz} & \dot{\sigma}_{xz} + (\sigma_{xx} - \sigma_{zz})\omega_{xz} \\ \dot{\sigma}_{xz} + (\sigma_{xx} - \sigma_{zz})\omega_{xz} & \dot{\sigma}_{zz} + 2\sigma_{xz}\omega_{xz} \end{bmatrix} \tag{2.29}$$

can be identified as the 'Jaumann' or 'co-rotational' time derivative of σ and is an objective quantity independent of superposed rigid body rotations. Hence (2.28) can be expressed as

$$d_{xx} = -d_{zz} = \frac{1}{2}\dot{\lambda}(\sigma_{xx} - \sigma_{zz}) - \frac{1}{4q}\sin\phi\,(\overset{\triangledown}{\sigma}_{xx} - \overset{\triangledown}{\sigma}_{zz}), \quad d_{xz} = \dot{\lambda}\sigma_{xz} - \frac{1}{2q}\sin\phi\overset{\triangledown}{\sigma}_{xz}, \tag{2.30}$$

which is a concise and convenient properly invariant formulation of the double-shearing equations that is consistent with material isotropy. This formulation and some extensions of it were given in [2]. An equation similar in form to (2.30) (but including elastic deformation) was proposed by Rudnicki and Rice [14] as a model for fissured rock masses, but their model is a hardening model and the interpretation differs from that of the double-shearing model.

Alternatively, using (2.4), we can express (2.29) as

$$\overset{\triangledown}{\sigma} = \begin{bmatrix} \overset{\triangledown}{\sigma}_{xx} & \overset{\triangledown}{\sigma}_{xz} \\ \overset{\triangledown}{\sigma}_{xz} & \overset{\triangledown}{\sigma}_{zz} \end{bmatrix} = \begin{bmatrix} \dot{\sigma}_{xx} & \dot{\sigma}_{xz} \\ \dot{\sigma}_{xz} & \dot{\sigma}_{zz} \end{bmatrix} + 2q\omega_{xz}\begin{bmatrix} -\sin 2\psi & \cos 2\psi \\ \cos 2\psi & \sin 2\psi \end{bmatrix} \tag{2.31}$$

and (2.30) as

$$d_{xx} = -d_{zz} = \dot{\lambda}q\cos 2\psi - \frac{1}{4q}\sin\phi\,(\overset{\triangledown}{\sigma}_{xx} - \overset{\triangledown}{\sigma}_{zz}), \quad d_{xz} = \dot{\lambda}q\sin 2\psi - \frac{1}{2q}\sin\phi\overset{\triangledown}{\sigma}_{xz}. \tag{2.32}$$

It is also useful to invert (2.30). We note the relation, that follows from (2.2) and (2.31)

$$(\sigma_{xx} - \sigma_{zz})(\overset{\nabla}{\sigma}_{xx} - \overset{\nabla}{\sigma}_{zz}) + 4\overset{\nabla}{\sigma}_{xz}\sigma_{xz} = 4\dot{q}q. \tag{2.33}$$

Hence it follows from (2.30) that

$$q\{(d_{xx} - d_{zz})\cos 2\psi + 2d_{xz}\sin 2\psi\} = 2\dot{\lambda}q^2 - \dot{q}\sin\phi. \tag{2.34}$$

The left-hand side of (2.34) represents the rate of working of the stress in the velocity field. It follows that

$$\dot{\lambda} = \frac{\{(d_{xx} - d_{zz})\cos 2\psi + 2d_{xz}\sin 2\psi\}}{2q} + \frac{\dot{q}\sin\phi}{2q^2} \tag{2.35}$$

and therefore (2.30) can be rearranged in the form

$$\begin{aligned} (\overset{\nabla}{\sigma}_{xx} - \overset{\nabla}{\sigma}_{zz}) &= -\frac{2q(d_{xx} - d_{zz})}{\sin\phi} + \left\{\frac{\{(d_{xx} - d_{zz})\cos 2\psi + 2d_{xz}\sin 2\psi\}}{q\sin\phi} + \frac{\dot{q}}{q}\right\}(\sigma_{xx} - \sigma_{zz}), \\ \overset{\nabla}{\sigma}_{xz} &= -\frac{2qd_{xz}}{\sin\phi} + \left\{\frac{\{(d_{xx} - d_{zz})\cos 2\psi + 2d_{xz}\sin 2\psi\}}{q\sin\phi} + \frac{\dot{q}}{q}\right\}\sigma_{xz}. \end{aligned} \tag{2.36}$$

We note also that

$$\overset{\nabla}{\sigma}_{xx} = \frac{1}{2}(\overset{\nabla}{\sigma}_{xx} - \overset{\nabla}{\sigma}_{zz}) - \dot{p}, \quad \overset{\nabla}{\sigma}_{zz} = -\frac{1}{2}(\overset{\nabla}{\sigma}_{xx} - \overset{\nabla}{\sigma}_{zz}) - \dot{p}. \tag{2.37}$$

Equation (2.36) may be regarded as a form of hypoplastic constitutive equation. Hypoplastic equations are constitutive equations of the general form (in direct notation) $\overset{\nabla}{\sigma} = \mathbf{F}(\sigma, \mathbf{D})$, where $\mathbf{F}$ is an isotropic tensor function of σ and $\mathbf{D}$ that is homogeneous of degree one in the rate-of-deformation $\mathbf{D}$. Equation (2.36) is clearly a special case of this relation. The interpretation of the double-shearing theory as a hypoplastic constitutive equation was pointed out by Mehrabadi and Cowin [11]. Hypoplastic theories of granular materials have been studied extensively by Kolymbas, Wu and collaborators (see, for example [17] which contains references to earlier developments).

It has been pointed out by several authors (for example Harris [12]) that the equations of the double-shearing theory (and of several other theories for the mechanics of granular materials) are linearly ill-posed with respect to initial conditions. Schaeffer [13] has suggested connections between ill-posedness and the formation of shear bands, which are commonly observed in real granular material, but this is still a matter for debate.

3. Simple shear flow

The problem of steady simple shear flow of a layer of granular material was analyzed by Spencer [19]. It was shown there that there exists a solution in which the stress is constant, but that this solution is unstable and there is an alternative solution in which the stress is time-dependent. Details of the solution were obtained under the boundary condition that the lateral confining pressure is constant. For that case it was shown that in the time-dependent solution the shear stress required to support the deformation increases initially, but then passes through a maximum and then decreases as the shear strain increases. In this section we consider the same deformation but with a different boundary condition, namely we suppose that a constant normal pressure P acts on the shear planes. We also extend the solution to include inertia terms.

Thus it is supposed that the material is subject to a pressure P in the z direction (as, for example, in a layer of granular material with a uniform rigid overburden) so that $\sigma_{zz} = -P$ uniformly. Hence from (2.4)

$$P = p + q\cos 2\psi \tag{3.1}$$

and so, from (2.5)

$$p = \frac{P - c\cos\phi\cos 2\psi}{1+\sin\phi\cos 2\psi}, \quad q = \frac{P\sin\phi + c\cos\phi}{1+\sin\phi\cos 2\psi}, \tag{3.2}$$

and from (2.4)

$$\begin{aligned} \sigma_{xx} &= \frac{-P(1-\sin\phi\cos 2\psi) + 2c\cos\phi\cos 2\psi}{1+\sin\phi\cos 2\psi}, \\ \sigma_{zz} &= -P, \quad \sigma_{xz} = \frac{(P\sin\phi + c\cos\phi)\sin 2\psi}{1+\sin\phi\cos 2\psi}. \end{aligned} \tag{3.3}$$

We consider the simple shearing deformation

$$u = \alpha z, \quad w = 0, \tag{3.4}$$

where α is a positive constant. It is assumed that the stress, and hence ψ, is independent of position, but may depend on time t. Then the isochoric condition (2.18) is satisfied, and the double-shearing condition (2.20) becomes

$$\alpha(\cos 2\psi + \sin\phi) + 2\frac{\mathrm{d}\psi}{\mathrm{d}t}\sin\phi = 0. \tag{3.5}$$

3.1. Steady stress solution

Equation (3.5) clearly has the solution

$$\cos 2\psi = -\sin\phi, \tag{3.6}$$

so that

$$\psi = \psi_S = \frac{1}{4}\pi + \frac{1}{2}\phi \tag{3.7}$$

and the positive sign for ψ_s has been chosen to ensure that the plastic work-rate is positive. Correspondingly, the stress is

$$\sigma_{xx} = -\frac{P(1+\sin^2\phi)}{\cos^2\phi} - 2c\tan\phi, \quad \sigma_{zz} - -P, \quad \sigma_{xz} = P\tan\phi + c. \tag{3.8}$$

Thus in this solution the stress is independent of t and the material undergoes a steady shearing flow with a constant shear stress.

To investigate the linear stability of this time-independent stress solution we introduce a small perturbation

$$\psi = \psi_S + \varepsilon\psi_1, \quad u = \alpha y + \varepsilon u_1, \quad v = \varepsilon v_1, \tag{3.9}$$

where ε is a small parameter. We suppose that the perturbations ψ_1, u_1, v_1 depend only on t and are independent of position (this is not the most general case but is sufficient for our purpose). The substitution of (3.9) in (3.5) gives, to first order in ε,

$$\frac{\mathrm{d}\psi_1}{\mathrm{d}t} - \alpha\psi_1 \cot\phi = 0 \tag{3.10}$$

and hence

$$\psi_1 = A \exp(\alpha t \cot\phi), \tag{3.11}$$

where A is a constant. Therefore, since α is positive, small perturbations of the form (3.9) of the steady-stress solution grow exponentially, and the time-independent stress solution is linearly unstable.

We note in passing that for constant α, the solution (3.4), (3.6) and (3.8) also satisfies the dynamic equations, with inertia terms included. It can be shown by a small extension of the above argument that the time-independent stress solution is also unstable, in a similar way, when regarded as a solution of the dynamic equations. This is consistent with several exact solutions of the dynamic equations obtained by Hill and Spencer [22] which showed in all the cases considered that an increasing shear strain is accompanied by a decreasing shear stress. It is also consistent with observations that dynamic shear deformations in earthquakes and avalanches are often larger than is suggested by assuming a steady shear stress solution; this effect is observed even in fluid-free environments such as the moon.

3.2. Time-dependent stress solution

The above analysis shows that there is a bifurcation from the time-independent stress solution at $\psi = \psi_S$. To analyze the post-bifurcation behaviour we return to (3.5). It was shown in [19] that (3.5) has the solution, subject to the initial condition $\psi = \psi_0$ at $t = 0$,

$$\frac{\tan\psi}{\tan\psi_S} = \frac{\tan\psi_S \tanh(\frac{1}{2}\alpha t \cot\phi) - \tan\psi_0}{\tan\psi_0 \tanh(\frac{1}{2}\alpha t \cot\phi) - \tan\psi_S}. \tag{3.12}$$

This reduces to (3.7) if $\psi_0 = \psi_S$. However, if ψ_0 is not precisely equal to ψ_S, then ψ increases with t if $\psi_0 > \psi_S$ and decreases with t if $\psi_0 < \psi_S$. Thus, in either case the principal axes of stress rotate *away* from the directions defined by the time-independent solution (3.7). In the limit $t \to \infty$ (3.12) gives $\tan\psi \to -\tan\psi_S$, but negative value of ψ are not relevant because they imply negative plastic work rate.

From (3.3) and (3.7), the shear stress σ_{xz} can be expressed as

$$\sigma_{xz} = \frac{2(P\sin\phi + c\cos\phi)\tan\psi}{(1-\sin\phi)(\tan^2\psi_S + \tan^2\psi)}, \tag{3.13}$$

from which it is easily shown that σ_{xz} has its maximum value when $\psi = \psi_S$ and decreases monotonically as ψ increases or decreases from this value. This is in contrast to the behaviour described in [19] for the case in which shear takes place under constant confining stress σ_{xx}. In that case σ_{xz} initially increases as ψ decreases from the value ψ_S, then reaches a maximum value at $\psi = \frac{1}{4}\pi - \frac{1}{2}\phi$ at a finite shear strain whose value depends on ψ_0, and finally decreases, eventually to zero (also at a finite shear strain) as ψ decrease further. Thus it seems that details of the behaviour may be sensitive to the precise conditions under which the simple shear flow occurs, but that the flow will always become unstable at some stage.

3.3. Shear-band formation

Instability of homogeneous deformations is often interpreted as leading to the formation of shear bands, or strain localization. We follow the approach formulated by Rudnicki and Rice [14] and Rice [15], in which strain localization is regarded as an instability in an initially uniform stress and velocity field. References to earlier literature are given in [14] and [15]. There is a very extensive subsequent literature on the subject – the main difference between different investigations is in the constitutive equation adopted. Some representative references are [17,18,21,22]. As far as we are aware shear band formation has not been studied explicitly in relation to the double-shearing theory, although related theories such as the yield vertex model of Rudnicki and Rice [14] and various hypoplastic models (for example [17,18]) have been used in this context.

In this section we are only concerned with plane strain deformations. Suppose that initially a body deforms with uniform velocity and stress fields $\mathbf{v}_0$ and σ_0. We seek to determine if there can exist an additional superposed field, with velocity $\mathbf{v}_1$ and stress σ_1, confined to a narrow band with the body. It is assumed that the velocity is continuous across the boundary surfaces of the band, but that the velocity gradients may be discontinuous. It follows that within the band the superposed velocity gradients are of the form

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} g_1 n_1 & g_1 n_2 \\ g_2 n_1 & g_2 n_2 \end{bmatrix}, \tag{3.14}$$

where

$$\mathbf{n} = (n_1, n_2) \text{ and } \mathbf{g} = (g_1, g_2) \tag{3.15}$$

are, respectively, the unit normal to the shear band and an arbitrary vector. It follows from the incompressibility condition that

$$g_1 n_1 + g_2 n_2 = 0. \tag{3.16}$$

It is also assumed that equilibrium is maintained during the formation of the shear band. The conditions for continuing equilibrium across the shear band is

$$\begin{bmatrix} \left|\dot{\sigma}_{xx}\right| & \left|\dot{\sigma}_{xz}\right| \\ \left|\dot{\sigma}_{xz}\right| & \left|\dot{\sigma}_{zz}\right| \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3.17}$$

where $\left|\dot{\sigma}\right|$ denotes the jump across the shear band. From (2.29) it follows that

$$\begin{bmatrix} \left|\overset{\nabla}{\sigma}_{xx} + 2\sigma_{xz}\omega_{xz}\right| & \left|\overset{\nabla}{\sigma}_{xz} - (\sigma_{xx} - \sigma_{zz})\omega_{xz}\right| \\ \left|\overset{\nabla}{\sigma}_{xz} - (\sigma_{xx} - \sigma_{zz})\omega_{xz}\right| & \left|\overset{\nabla}{\sigma}_{zz} - 2\sigma_{xz}\omega_{xz}\right| \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.18}$$

From (3.14) we have

$$(d_{xx},\ d_{zz},\ d_{xz}) = \left(g_1 n_1,\ g_2 n_2,\ \frac{1}{2}(g_1 n_2 + g_2 n_1)\right), \quad \omega_{xz} = \frac{1}{2}(g_1 n_2 - g_2 n_1) \tag{3.19}$$

and from (2.37) and (3.18)

$$\begin{aligned} n_1\{\frac{1}{2}(\overset{\nabla}{\sigma}_{xx} - \overset{\nabla}{\sigma}_{zz}) + 2\sigma_{xz}\omega_{xz}\} + n_2\{\overset{\nabla}{\sigma}_{xz} - (\sigma_{xx} - \sigma_{zz})\omega_{xz}\} - n_1\dot{p} &= 0, \\ n_1\{\overset{\nabla}{\sigma}_{xz} - (\sigma_{xx} - \sigma_{zz})\omega_{xz}\} - n_2\{\frac{1}{2}(\overset{\nabla}{\sigma}_{xx} - \overset{\nabla}{\sigma}_{zz}) + 2\sigma_{xz}\omega_{xz}\} - n_2\dot{p} &= 0. \end{aligned} \tag{3.20}$$

Hence there follows, since **n** is a unit vector,

$$\begin{aligned} &\frac{1}{2}(\overset{\nabla}{\sigma}_{xx} - \overset{\nabla}{\sigma}_{zz}) + 2\sigma_{xz}\omega_{xz} - (n_1^2 - n_2^2)\dot{p} = 0, \\ &\overset{\nabla}{\sigma}_{xz} - (\sigma_{xx} - \sigma_{zz})\omega_{xz}\} - 2n_1 n_2\dot{p} = 0. \end{aligned} \tag{3.21}$$

By substituting from (2.36) in (3.21), using (2.4) and noting that $\dot{q} = \dot{p}\sin\phi$, we may deduce that

$$\begin{aligned} &\sin 2\psi\{-(d_{xx} - d_{zz})\sin 2\psi + 2d_{xz}\cos 2\psi - 2\omega_{xz}\sin\phi\} + \frac{\dot{q}}{q}\{\sin\phi\cos 2\psi - (n_1^2 - n_2^2)\} = 0, \\ &\cos 2\psi\{-(d_{xx} - d_{zz})\sin 2\psi + 2d_{xz}\cos 2\psi - 2\omega_{xz}\sin\phi\} - \frac{\dot{q}}{q}\{\sin\phi\sin 2\psi - 2n_1 n_2\} = 0. \end{aligned} \tag{3.22}$$

In order for bifurcation to be possible, (3.16), (3.19) and (3.22) must admit non-trivial solutions for g_1, g_2 and $\dot{q}$. From (3.22) there follows

$$\frac{\dot{q}}{q}\{\sin\phi - (n_1^2 - n_2^2)\cos 2\psi - 2n_1 n_2\sin 2\psi\} = 0, \tag{3.23}$$

and hence either $\dot{q} = 0$, or

$$\sin\phi - (n_1^2 - n_2^2)\cos 2\psi - 2n_1 n_2\sin 2\psi = 0. \tag{3.24}$$

If $\dot{q} = 0$, then from (3.22)

$$-(d_{xx} - d_{zz})\sin 2\psi + 2d_{xz}\cos 2\psi - 2\omega_{xz}\sin\phi = 0. \tag{3.25}$$

From (3.16) and (3.19), this implies that

$$-2n_1 n_2\sin 2\psi - (n_1^2 - n_2^2)\cos 2\psi + \sin\phi = 0, \tag{3.26}$$

which is the same as (3.24). We define the angle δ as

$$\tan\delta = \frac{n_2}{n_1} \tag{3.27}$$

so that δ is the angle the normal to the shear band makes with the x-axis. Then (3.24) and (3.26) can be expressed as

$$-\cos 2(\psi - \delta) + \sin\phi = 0 \tag{3.28}$$

from which it follows that

$$\delta = \psi \pm (\frac{1}{4}\pi - \frac{1}{2}\phi). \tag{3.29}$$

Thus in either case $\dot{q} = 0$ or $\dot{q} \neq 0$, the normal to the shear band makes an angle $\pm(\frac{1}{4}\pi - \frac{1}{2}\phi)$ with the major principal stress axis. This means that the shear band is inclined at $\pm(\frac{1}{4}\pi + \frac{1}{2}\phi)$ to the major principal stress axis. Thus, from (2.9), a shear band is necessarily a line on which the critical shear stress is mobilized.

If $\dot{q} \neq 0$, then from (3.22)

$$\begin{aligned}&\sin 2\psi\{-(d_{xx} - d_{zz})\sin 2\psi + 2d_{xz}\cos 2\psi - 2\omega_{xz}\sin\phi\}\{\sin\phi\sin 2\psi - 2n_1n_2\} \\ &+ \cos 2\psi\{-(d_{xx} - d_{zz})\sin 2\psi + 2d_{xz}\cos 2\psi - 2\omega_{xz}\sin\phi\}\{\sin\phi\cos 2\psi - (n_1^2 - n_2^2)\} = 0\end{aligned} \tag{3.30}$$

which simplifies to

$$\{-(d_{xx} - d_{zz})\sin 2\psi + 2d_{xz}\cos 2\psi - 2\omega_{xz}\sin\phi\}\{\sin\phi - (n_1^2 - n_2^2)\cos 2\psi - 2n_1n_2\sin 2\psi\} = 0. \tag{3.31}$$

Both factors of (3.31) again yield (3.29), and so this case gives no additional information.

In the case of simple shear as considered in this section, in the steady stress solution we have

$$\psi = \psi_S = \frac{1}{4}\pi + \frac{1}{2}\phi,$$

and hence $\delta = \frac{1}{2}\pi$ or $\delta = \phi$. Therefore, any shear band is either parallel to the shear planes $z =$ const or the normal to shear band makes the angle ϕ with the shear planes. Many experimental studies of shear bands in simple shear (for example [23]) show shear bands approximately parallel to the shear planes.

Dilatancy probably plays a part in shear-band formation in real granular materials. For simplicity, we have not incorporated dilatant behaviour in this analysis, but it should be straightforward to extend the analysis to, for example, the dilatant double-shearing theory proposed by Mehrabadi and Cowin [9, 10].

4. Pure shear

A pure shearing deformation is the plane deformation defined by the velocity field

$$u = ex, \quad w = -ez, \tag{4.1}$$

where e is a constant which for definiteness (and without loss of generality) is taken to be positive. This field trivially satisfies the condition (2.18) for the deformation to be isochoric. The double-shearing condition (2.20) reduces to

$$-e\sin 2\psi + \sin\phi\frac{d\psi}{dt} = 0. \tag{4.2}$$

4.1. Time-independent stress solution

With the equilibrium equations (or equations of motion when e is constant), (4.2) has the solution

$$\psi = 0, \quad p = \frac{1}{2}(X + Z), \quad q = \frac{1}{2}(Z - X), \tag{4.3}$$

which corresponds to the constant stress

$$\sigma_{xx} = -X, \quad \sigma_{zz} = -Z, \quad \sigma_{xz} = 0. \tag{4.4}$$

The Coulomb yield condition (2.5) is satisfied as an equality provided that

$$Z = \frac{X(1 + \sin\phi) + 2c\cos\phi}{1 - \sin\phi}. \tag{4.5}$$

Hence (4.1)-(4.5) determine an exact time-independent stress solution.

To examine the linear stability of this solution, we introduce a perturbation from the time-independent solution $\psi = 0$ by setting $\psi = \varepsilon\psi_1$ in (4.2) and retaining only terms linear in ε. This gives

$$\frac{\mathrm{d}\psi_1}{\mathrm{d}t} = 2e\psi_1 \operatorname{cosec}\phi \tag{4.6}$$

which has the solution

$$\psi_1 = \psi_0 \exp(2et \operatorname{cosec}\phi), \tag{4.7}$$

where $\psi = \psi_0$ at $t = 0$, Hence in this case also, any variation from the initial value $\psi = 0$ (that is, if ψ_0 has any value other than zero) results in an exponentially growing deviation from the time-independent solution (4.3), and so the time-independent pure shear solution is also unstable.

4.2. Time-dependent stress solution

When $\psi_0 \neq 0$, (4.2) can be integrated to give the time-dependent stress solution

$$\tan\psi = \tan\psi_0 \exp(2et \operatorname{cosec}\phi). \tag{4.8}$$

Thus in pure shear also, unless $\psi_0 = 0$, the principal axes rotate away from the directions they assume in the time-independent solution, and $\psi \to \pm\frac{1}{2}\pi$ (according to whether ψ_0 is positive or negative), as $t \to \infty$. In practice, for pure shear of a finite body, it might be possible to stabilize the body by applying suitable tractions at the surface. For example, if the body is bounded by perfectly smooth surfaces at $x = \pm a$, then this enforces $\sigma_{xz} = 0$ at $x = \pm a$, which is incompatible with (4.8) unless $\psi_0 = 0$.

For a fixed lateral pressure X, the vertical compressive pressure $Z(t) = -\sigma_{zz}$ is given as

$$Z(t) = \frac{2(X\sin\phi + c)}{1 - \sin\phi\cos 2\psi} = \frac{2(X\sin\phi + c)(1 + \tan^2\psi)}{(1 - \sin\phi) + (1 + \sin\phi)\tan^2\psi}. \tag{4.9}$$

Hence $Z(t)$ is maximum at $\psi = 0$ and decreases as ψ either increases or decreases, confirming that the motion is unstable if the σ_{zz} stress is maintained. However, in contrast to the simple shear case, $Z(t)$ remains finite and bounded, and

$$Z(t) \to \frac{2(X\sin\phi + c)}{1 + \sin\phi} \quad \text{as } t \to \pm\infty. \tag{4.10}$$

If the body is finite and surface *forces* (rather than tractions) are prescribed, then geometrical changes may also influence the stability and instability of the deformation.

4.3. SHEAR-BAND FORMATION

The shear-band analysis presented in Section 3.3 applies generally for plane strain deformations and can equally be applied to the pure shear problem. Just as in that section the time-independent stress solution may bifurcate and a shear band may form whose normal direction is defined by the angle δ given by (3.29). In the case of pure shear, in the time-independent solution we have $\psi = 0$, so that

$$\delta = \pm(\frac{1}{4}\pi - \frac{1}{2}\phi) \tag{4.11}$$

and therefore the shear bands are inclined at $\pm(\frac{1}{4}\pi + \frac{1}{2}\phi)$ to the major principal stress direction. This is in good agreement with experimental observation. Shear bands at angles close to $\pm(\frac{1}{4}\pi + \frac{1}{2}\phi)$ are frequently observed in biaxial compression tests of granular materials (see, for example [24]).

5. Axially symmetric stress and deformation

Now consider that the stress and deformation fields are axially symmetric. For this case we refer vector and tensor quantities to cylindrical polar coordinates (r, θ, z), with the z axis as the axis of symmetry. In axial symmetry, stress and velocity components are independent of the polar angle θ. In (r, θ, z) coordinates, components of the stress tensor σ are denoted as

$$\sigma = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{rz} & \sigma_{\theta z} & \sigma_{zz} \end{bmatrix}, \tag{5.1}$$

and in axial symmetry $\sigma_{r\theta} = 0$ and $\sigma_{\theta z} = 0$. We consider only quasi-static deformations with no body forces, so that the stress satisfies the equilibrium equations which reduce to

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0. \tag{5.2}$$

We adopt the Coulomb-Mohr criterion in the form

$$\sigma_I - \sigma_{III} \leq 2c\cos\phi - (\sigma_I + \sigma_{III})\sin\phi, \tag{5.3}$$

where $\sigma_I, \sigma_{II}, \sigma_{III}$ are principal components of stress, ordered so that $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$. In axial symmetry $\sigma_{\theta\theta}$ is always one of the principal stress components. Various possibilities arise, depending on whether or not the principal stress components are distinct and which of σ_I, σ_{II} or σ_{III} coincides with $\sigma_{\theta\theta}$. The various stress regimes were classified by Cox, Eason and Hopkins [25], and Spencer [2,26] also gives an account of them. It seems that particular significance is attached to the so-called ‘Haar-von Karman’ regime in which $\sigma_{\theta\theta}$ is equal to one of the principal stress components associated with a principal direction in the (r, z) plane. The case of interest here is that in which

$$\sigma_I = \sigma_{\theta\theta} > \sigma_{III} \tag{5.4}$$

and, when the material is undergoing flow

$$\sigma_I(1+\sin\phi) = \sigma_{III}(1-\sin\phi) + 2c\cos\phi. \tag{5.5}$$

In this case the stress can be expressed as

$$\sigma_{rr} = -p + q\cos 2\widetilde{\psi}, \quad \sigma_{zz} = -p - q\cos 2\widetilde{\psi}, \quad \sigma_{rz} = q\sin 2\widetilde{\psi}, \quad \sigma_{\theta\theta} = -p + q, \tag{5.6}$$

where

$$\begin{aligned} p &= -\frac{1}{2}(\sigma_I + \sigma_{III}) = -\frac{1}{2}(\sigma_{rr} + \sigma_{zz}), \\ q &= \frac{1}{2}(\sigma_I - \sigma_{III}) = \left\{\frac{1}{4}(\sigma_{rr} - \sigma_{zz})^2 + \sigma_{rz}^2\right\}^{\frac{1}{2}}, \quad \tan 2\widetilde{\psi} = \frac{2\sigma_{rz}}{\sigma_{rr} - \sigma_{zz}}, \end{aligned} \tag{5.7}$$

so that $\widetilde{\psi}$ represents the angle that the algebraically greater principal stress direction in the (r, z) planes makes with the r-direction, and the Coulomb -Mohr yield condition takes the form

$$q = p\sin\phi + c\sin\phi, \tag{5.8}$$

which, of course, is formally the same as the corresponding condition (2.5) in the plane strain case. The derivation of (5.8) is similar to the derivation of (2.5), and just as in the plane strain case it can be shown that the critical stress is mobilized on the surfaces

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \tan\left(\widetilde{\psi} \pm (\frac{1}{4}\pi + \frac{1}{2}\phi)\right). \tag{5.9}$$

In the quasi-static case, (5.2), (5.6) and (5.8) can be reduced to two first-order partial differential equations for q and $\widetilde{\psi}$: these equations are hyperbolic, with characteristics defined by (5.9). In the dynamic case acceleration terms must be included and the stress solution does not uncouple from the velocity field.

When $\sigma_{\theta\theta} = \sigma_I$, and (5.5) is satisfied, the critical shear stress at a generic point (r_0, z_0) is mobilized on all surfaces whose normals make angles $\frac{1}{4}\pi + \frac{1}{2}\phi$ with the principal stress direction associated with the principal stress σ_{III}. The unit vector that characterizes this direction is denoted by $\mathbf{e}_3$. Following arguments similar to those used in Section 2, it is assumed that the flow is a superposition of shear flows on all of these possible shear surfaces. In addition, since the other two principal directions are not uniquely defined, an arbitrary spin about the $\mathbf{e}_3$ direction may also be superposed. Details of the development were given in [26]. The result is that the velocity field is governed by the equations

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0, \tag{5.10}$$

$$\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)\cos 2\widetilde{\psi} - \left(\frac{\partial v_r}{\partial r} - \frac{\partial v_z}{\partial z}\right)\sin 2\widetilde{\psi} + \sin\phi\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} + 2\widetilde{\Omega}\right) = 0, \tag{5.11}$$

where v_r, v_z denote components of velocity in the (r, z) plane, and

$$\widetilde{\Omega} = \dot{\widetilde{\psi}} = \frac{\partial\widetilde{\psi}}{\partial t} + v_r\frac{\partial\widetilde{\psi}}{\partial r} + v_z\frac{\partial\widetilde{\psi}}{\partial z} \tag{5.12}$$

is the spin of the principal axes of stress through a generic particle. The similarity between (5.10–5.12) on the one hand and (2.18), (2.20) and (2.21) on the other is evident. Equation (5.10) states that the flow is isochoric, and (5.11) is of exactly the same form as (2.20) and expresses the double shearing (or, more accurately, multi-shearing) deformation mechanism in axially symmetric deformations.

By arguments that directly parallel those of Section 2, we deduce from (5.11) that

$$\begin{aligned} d_{rr} - d_{zz} &= \dot{\lambda}(\sigma_{rr} - \sigma_{zz}) - \frac{1}{2q}\sin\phi\{(\dot{\sigma}_{rr} - \dot{\sigma}_{zz}) - 4\sigma_{rz}\omega_{rz}\}, \\ d_{rz} &= \dot{\lambda}\sigma_{rz} - \frac{1}{2q}\sin\phi\{\dot{\sigma}_{rz} + (\sigma_{rr} - \sigma_{zz})\omega_{rz}\}, \end{aligned} \tag{5.13}$$

where

$$(d_{rr}, d_{\theta\theta}, d_{zz}, 2d_{rz}) = \left(\frac{\partial v_r}{\partial r}, \frac{v_r}{r}, \frac{\partial v_z}{\partial z}, \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right), \quad 2\omega_{rz} = -2\omega_{zr} = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}. \tag{5.14}$$

Furthermore, the Jaumann derivatives of σ_{rr}, σ_{zz}, and σ_{rz} are

$$\overset{\triangledown}{\sigma}_{rr} = \dot{\sigma}_{rr} - 2\omega_{rz}\sigma_{rz}, \quad \overset{\triangledown}{\sigma}_{zz} = \dot{\sigma}_{zz} + 2\omega_{rz}\sigma_{rz}, \quad \overset{\triangledown}{\sigma}_{rz} = \dot{\sigma}_{rz} + \omega_{rz}(\sigma_{rr} - \sigma_{zz}), \tag{5.15}$$

and therefore (5.13) can be written as

$$d_{rr} - d_{zz} = \dot{\lambda}(\sigma_{rr} - \sigma_{zz}) - \frac{1}{2q}\sin\phi\,(\overset{\triangledown}{\sigma}_{rr} - \overset{\triangledown}{\sigma}_{zz}), \quad d_{rz} = \dot{\lambda}\sigma_{rz} - \frac{1}{2q}\sin\phi\overset{\triangledown}{\sigma}_{rz}. \tag{5.16}$$

It follows that

$$(d_{rr} - d_{zz})(\sigma_{rr} - \sigma_{zz}) + 4d_{rz}\sigma_{rz} = 4\dot{\lambda}q^2 - 2q\dot{q}\sin\phi \tag{5.17}$$

and hence

$$\dot{\lambda} = \frac{(d_{rr} - d_{zz})(\sigma_{rr} - \sigma_{zz}) + 4d_{rz}\sigma_{rz}}{4q^2} + \frac{\dot{q}\sin\phi}{2q^2}. \tag{5.18}$$

We now have, from (5.16) and (5.18)

$$\begin{aligned} \overset{\triangledown}{\sigma}_{rr} &= \left\{\frac{(d_{rr} - d_{zz})(\sigma_{rr} - \sigma_{zz}) + 4d_{rz}\sigma_{rz}}{4q^2\sin\phi} + \frac{\dot{q}}{2q}\right\}(\sigma_{rr} - \sigma_{zz}) - \frac{q}{\sin\phi}(d_{rr} - d_{zz}) - \dot{p}, \\ \overset{\triangledown}{\sigma}_{zz} &= \left\{\frac{(d_{rr} - d_{zz})(\sigma_{rr} - \sigma_{zz}) + 4d_{rz}\sigma_{rz}}{4q\sin\phi} + \frac{\dot{q}}{2q}\right\}(\sigma_{zz} - \sigma_{rr}) - \frac{q}{\sin\phi}(d_{zz} - d_{rr}) - \dot{p}, \\ \overset{\triangledown}{\sigma}_{rz} &= \left\{\frac{(d_{rr} - d_{zz})(\sigma_{rr} - \sigma_{zz}) + 4d_{rz}\sigma_{rz}}{2q\sin\phi} + \frac{\dot{q}}{q}\right\}\sigma_{rx} - \frac{2q}{\sin\phi}d_{rz}. \end{aligned} \tag{5.19}$$

6. Compression of a circular cylinder

This is the axially symmetric analogue of pure shear. A circular cylinder of granular material is compressed by axial forces, while confined by a radial pressure. In soil mechanics literature, the experiment is often referred to as the triaxial test. The deformation is described by the homogeneous velocity field

$$v_r = \frac{1}{2}kr, \quad v_z = -kz \tag{6.1}$$

where k is constant. This satisfies the condition (5.10) for the motion to be isochoric. The double-shearing relation (5.11) reduces to

$$-\frac{3}{2}k \sin 2\widetilde{\psi} + 2\frac{d\widetilde{\psi}}{dt} \sin\phi = 0. \tag{6.2}$$

6.1. Time-independent stress solution

Together with the equilibrium equations, (6.2) has the time-independent stress solution

$$\sigma_{rr} = -R, \quad \sigma_{\theta\theta} = -R, \quad \sigma_{zz} = -Z, \quad \sigma_{rz} = 0, \tag{6.3}$$

which corresponds to

$$\widetilde{\psi} = 0, \quad p = \frac{1}{2}(R+Z), \quad q = \frac{1}{2}(Z-R). \tag{6.4}$$

In order to satisfy the Coulomb-Mohr yield condition we have, in the time-independent solution

$$Z = \frac{R(1+\sin\phi) + 2c\cos\phi}{1-\sin\phi}. \tag{6.5}$$

Exactly as in Section 4, it follows that this solution is linearly unstable to perturbations of the form $\widetilde{\psi} = \varepsilon\widetilde{\psi}_1$.

6.2. Time-dependent stress solution

Also as in Section 4.2, (6.2) can be integrated to give the time-dependent exact solution

$$\tan\widetilde{\psi} = \tan\widetilde{\psi}_0 \exp\left(\frac{3}{2}kt\right) \operatorname{cosec}\phi \tag{6.6}$$

and all the conclusions of Section 4 apply in this case also, provided only that e is replaced by $3k/4$. In the time-dependent solution, the axial compressive stress required to produce the deformation is

$$Z(t) = \frac{2(R\sin\phi + c)}{1 - \sin\phi\cos 2\widetilde{\psi}} = \frac{2(R\sin\phi + c)(1+\tan^2\widetilde{\psi})}{(1-\sin\phi) + (1+\sin\phi)\tan^2\widetilde{\psi}}. \tag{6.7}$$

6.3. Conical shear-band formation

In Section 4.1 it was shown that instability of the uniform pure shear deformation may take the form of strain localization in shear bands inclined at angles $\pm(\frac{1}{4}\pi + \frac{1}{2}\phi)$ to the x-axis. The axisymmetric analogue in the triaxial test is strain localization in the neighbourhood of conical surfaces in the material. For the Haar-von Karman regime the analysis exactly parallels that of Section 3. We look for an superposed field confined to a narrow band in the neighbourhood of a conical surface with its apex on the z-axis and unit normal $\widetilde{\mathbf{n}} = (\widetilde{n}_1, \widetilde{n}_2) = (\cos\widetilde{\delta}, \sin\widetilde{\delta})$. The underlying velocity field is the uniform field (6.1). It follows that if the velocity is continuous across the conical band, but velocity gradients may be discontinuous, the superposed velocity gradients are of the form (analogous to (3.14))

$$\begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{\partial v_z}{\partial z} \end{bmatrix} = \begin{bmatrix} \widetilde{g}_1\widetilde{n}_1 & \widetilde{g}_1\widetilde{n}_2 \\ \widetilde{g}_2\widetilde{n}_1 & \widetilde{g}_2\widetilde{n}_2 \end{bmatrix}, \tag{6.8}$$

where $(\widetilde{g}_1, \widetilde{g}_{2)}$ is an arbitrary vector. The condition for the deformation to be isochoric is $\widetilde{g}_1\widetilde{n}_1 + \widetilde{g}_2\widetilde{n}_2$ =0 (the superposed deformation-rate component $d_{\theta\theta}$ is zero) and that for continuing equilibrium across the band is

$$\begin{bmatrix} \left|\dot{\sigma}_{rr}\right| & \left|\dot{\sigma}_{rz}\right| \\ \left|\dot{\sigma}_{rz}\right| & \left|\dot{\sigma}_{zz}\right| \end{bmatrix} \begin{bmatrix} \widetilde{n}_1 \\ \widetilde{n}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.9}$$

and hence

$$\begin{bmatrix} \left|\overset{\nabla}{\sigma}_{rr} + 2\sigma_{rz}\omega_{rz}\right| & \left|\overset{\nabla}{\sigma}_{rz} - (\sigma_{rr} - \sigma_{zz})\omega_{rz}\right| \\ \left|\overset{\nabla}{\sigma}_{rz} - (\sigma_{rr} - \sigma_{zz})\omega_{rz}\right| & \left|\overset{\nabla}{\sigma}_{zz} - 2\sigma_{rz}\omega_{rz}\right| \end{bmatrix} \begin{bmatrix} \widetilde{n}_1 \\ \widetilde{n}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6.10}$$

From this it follows, as in the argument that leads to (3.21), that

$$\frac{1}{2}(\overset{\nabla}{\sigma}_{rr} - \overset{\nabla}{\sigma}_{zz}) + 2\sigma_{rz}\omega_{rz} - (\widetilde{n}_1^2 - \widetilde{n}_2^2)\dot{p} = 0, \quad \overset{\nabla}{\sigma}_{rz} - (\sigma_{rr} - \sigma_{zz})\omega_{rz}\} - 2\widetilde{n}_1\widetilde{n}_2\dot{p} = 0. \tag{6.11}$$

Then, exactly as in Section 3, we find that non-trivial solutions for $\widetilde{g}_{1,}\widetilde{g}_2$ and $\dot{q}$ are admitted only if

$$\sin\phi - (\widetilde{n}_1^2 - \widetilde{n}_2^2)\cos 2\widetilde{\psi} - 2\widetilde{n}_1\widetilde{n}_2 \sin 2\widetilde{\psi} = 0, \tag{6.12}$$

which implies that

$$\widetilde{\delta} = \widetilde{\psi} \pm (\frac{1}{4}\pi - \frac{1}{2}\phi). \tag{6.13}$$

However, in the steady stress solution, $\widetilde{\psi} = 0$, and so in this case

$$\widetilde{\delta} = \pm(\frac{1}{4}\pi - \frac{1}{2}\phi). \tag{6.14}$$

Hence there exists the possibility of the formation of either upright or inverted conical shear bands of vertex semi-angle $\frac{1}{4}\pi - \frac{1}{2}\phi$. These conical surfaces coincide with the surfaces on which the critical stress stress is mobilized.

Strain localization in a circular cylindrical body under compression need not necessarily occur in an axially symmetric manner, and there is also a possibility of the formation of plane shear bands.

7. Conclusion

It has been shown that the the non-dilatant double-shearing theory of mechanics of granular materials admits steady stress solutions for the plane strain deformations of simple shear and pure shear, and for compression of a circular cylinder, but that these solutions are all linearly unstable. The theory also yields time-dependent exact solutions for these problems in which the deformation takes place under decreasing load, which also indicates instability. The theory has also been applied to an analysis of strain localization in these same deformations. This analysis gives predictions of shear band formation that are in broad agreement with observed behaviour in real dry free-flowing granular materials. The inclusion of dilatant and elastic response may be expected to moderate and refine these predictions.

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